FP1
Worksheet on Euler's Method: Solutions

1. **Given the differential equation $\frac{dy}{dx} = 2x - y$ and $y(0) = 2$, find an approximation for $y(1)$ using Euler's method with two equal steps.**
   Solution:
   The step size is $\Delta x = \frac{1-0}{2} = 0.5$. We start at $(x_0, y_0) = (0, 2)$.

$$
\begin{aligned}
x_0 &= 0, \quad y_0 = 2 \\
x_1 &= 0.5, \quad y_1 = y_0 + \Delta x \cdot f(x_0, y_0) = 2 + 0.5(2(0) - 2) = 2 + 0.5(-2) = 1 \\
x_2 &= 1.0, \quad y_2 = y_1 + \Delta x \cdot f(x_1, y_1) = 1 + 0.5(2(0.5) - 1) = 1 + 0.5(1 - 1) = 1
\end{aligned}
$$

   The approximation is $y(1) \approx 1$.

2. **Given the differential equation $\frac{dy}{dx} = 3x^2$ and $y = 2$ when $x = 1$, find an approximation for y when $x = 2$ by using Euler's method with two equal steps.**
   Solution:
   The step size is $\Delta x = \frac{2-1}{2} = 0.5$. We start at $(x_0, y_0) = (1, 2)$.

$$
\begin{aligned}
x_0 &= 1.0, \quad y_0 = 2 \\
x_1 &= 1.5, \quad y_1 = y_0 + \Delta x \cdot f(x_0, y_0) = 2 + 0.5(3(1)^2) = 2 + 1.5 = 3.5 \\
x_2 &= 2.0, \quad y_2 = y_1 + \Delta x \cdot f(x_1, y_1) = 3.5 + 0.5(3(1.5)^2) = 3.5 + 0.5(6.75) = 3.5 + 3.375 = 6.875
\end{aligned}
$$

   The approximation is $y(2) \approx 6.875$.

3. **A continuous function f and its derivative f' have values according to the table below. Use Euler's method to find the value of $f(2)$.**
   Solution:
   From the table, the step size is $\Delta x = 0.5$. We are given $x_0 = 1.0$, $y_0 = 3$, and $f'(1.0) = 2.0$.

$$
\begin{aligned}
x_0 &= 1.0, \quad y_0 = 3 \\
x_1 &= 1.5, \quad y_1 = y_0 + \Delta x \cdot f'(x_0) = 3 + 0.5(2.0) = 4 \\
x_2 &= 2.0, \quad y_2 = y_1 + \Delta x \cdot f'(x_1) = 4 + 0.5(2.5) = 4 + 1.25 = 5.25
\end{aligned}
$$

   The approximation is $f(2) \approx 5.25$.

4. **Given the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and $y(2) = 1$, find an approximation of $y(3)$ using Euler's method and $\Delta x = 0.5$.**
   Solution:
   We start at $(x_0, y_0) = (2, 1)$ and take steps of $\Delta x = 0.5$.

$$
\begin{aligned}
x_0 &= 2.0, \quad y_0 = 1 \\
x_1 &= 2.5, \quad y_1 = y_0 + \Delta x \cdot \left(\frac{x_0}{y_0}\right) = 1 + 0.5\left(\frac{2}{1}\right) = 1 + 1 = 2 \\
x_2 &= 3.0, \quad y_2 = y_1 + \Delta x \cdot \left(\frac{x_1}{y_1}\right) = 2 + 0.5\left(\frac{2.5}{2}\right) = 2 + 0.5(1.25) = 2 + 0.625 = 2.625
\end{aligned}
$$

   The approximation is $y(3) \approx 2.625$.

5. Given $\frac{dy}{dx} = x - 2y$ and $y(1) = 0$, find an approximation of $y(2)$ using two equal steps in Euler's method.

Solution:

The step size is $\Delta x = \frac{2-1}{2} = 0.5$. We start at $(x_0, y_0) = (1, 0)$.

$$
\begin{aligned}
x_0 &= 1.0, \quad y_0 = 0 \\
x_1 &= 1.5, \quad y_1 = y_0 + \Delta x \cdot f(x_0, y_0) = 0 + 0.5(1 - 2(0)) = 0 + 0.5(1) = 0.5 \\
x_2 &= 2.0, \quad y_2 = y_1 + \Delta x \cdot f(x_1, y_1) = 0.5 + 0.5(1.5 - 2(0.5)) = 0.5 + 0.5(0.5) = 0.75
\end{aligned}
$$

The approximation is $y(2) \approx 0.75$.

6. A function f and its derivative f' have values according to the table below. Use Euler's Method with two equal steps to approximate the value of $f(4.4)$.

Solution:

From the table, $\Delta x = 0.2$. We are given $x_0 = 4.0$, $y_0 = 2$, and $f'(4.0) = -0.5$.

$$
\begin{aligned}
x_0 &= 4.0, \quad y_0 = 2 \\
x_1 &= 4.2, \quad y_1 = y_0 + \Delta x \cdot f'(x_0) = 2 + 0.2(-0.5) = 2 - 0.1 = 1.9 \\
x_2 &= 4.4, \quad y_2 = y_1 + \Delta x \cdot f'(x_1) = 1.9 + 0.2(-0.3) = 1.9 - 0.06 = 1.84
\end{aligned}
$$

The approximation is $f(4.4) \approx 1.84$.

7. A curve passing through $(2, 0)$ satisfies the differential equation $\frac{dy}{dx} = 4x + y$. Find an approximation for $y(3)$ using Euler's method with two steps.
Solution:
The step size is $\Delta x = \frac{3-2}{2} = 0.5$. We start at $(x_0, y_0) = (2, 0)$.

$$\begin{aligned}
&x_0 = 2.0, \quad y_0 = 0 \\
&x_1 = 2.5, \quad y_1 = y_0 + \Delta x \cdot (4x_0 + y_0) = 0 + 0.5(4(2) + 0) = 0 + 0.5(8) = 4 \\
&x_2 = 3.0, \quad y_2 = y_1 + \Delta x \cdot (4x_1 + y_1) = 4 + 0.5(4(2.5) + 4) = 4 + 0.5(10 + 4) = 4 + 7 = 11
\end{aligned}$$

The approximation is $y(3) \approx 11$.

8. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = 2y + x$ with the initial condition $f(0) = 1$. Use Euler's Method starting at $x = 0$ with two steps of equal size to approximate $f(-0.6)$.
Solution:
The step size is $\Delta x = \frac{-0.6-0}{2} = -0.3$. We start at $(x_0, y_0) = (0, 1)$.

$$\begin{aligned}
&x_0 = 0, \quad y_0 = 1 \\
&x_1 = -0.3, \quad y_1 = y_0 + \Delta x \cdot (2y_0 + x_0) = 1 - 0.3(2(1) + 0) = 1 - 0.6 = 0.4 \\
&x_2 = -0.6, \quad y_2 = y_1 + \Delta x \cdot (2y_1 + x_1) = 0.4 - 0.3(2(0.4) - 0.3) = 0.4 - 0.3(0.5) = 0.4 - 0.15 = 0.25
\end{aligned}$$

The approximation is $f(-0.6) \approx 0.25$.

9. $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = \sqrt{e^x + 2e^y}$. Given that $f(2.5) = 0$, use five iterations of Euler's method to estimate the value of $f(3)$.
Solution:
The step size is $\Delta x = \frac{3-2.5}{5} = 0.1$. We start at $x_0 = 2.5$ and $y_0 = 0$. (A calculator is required for the intermediate steps. Results are rounded to four decimal places).

$$\begin{aligned}
&x_0 = 2.5, \quad y_0 = 0 \\
&x_1 = 2.6, \quad y_1 = y_0 + 0.1\sqrt{e^{2.5} + 2e^0} \approx 0 + 0.1\sqrt{12.1825 + 2} \approx 0.3766 \\
&x_2 = 2.7, \quad y_2 = y_1 + 0.1\sqrt{e^{2.6} + 2e^{y_1}} \approx 0.3766 + 0.1\sqrt{13.4637 + 2e^{0.3766}} \approx 0.7813 \\
&x_3 = 2.8, \quad y_3 = y_2 + 0.1\sqrt{e^{2.7} + 2e^{y_2}} \approx 0.7813 + 0.1\sqrt{14.8797 + 2e^{0.7813}} \approx 1.2200 \\
&x_4 = 2.9, \quad y_4 = y_3 + 0.1\sqrt{e^{2.8} + 2e^{y_3}} \approx 1.2200 + 0.1\sqrt{16.4446 + 2e^{1.2200}} \approx 1.7019 \\
&x_5 = 3.0, \quad y_5 = y_4 + 0.1\sqrt{e^{2.9} + 2e^{y_4}} \approx 1.7019 + 0.1\sqrt{18.1741 + 2e^{1.7019}} \approx 2.2417
\end{aligned}$$

The approximation is $f(3) \approx 2.242$.

10. The differential equation $\frac{dy}{dx} = \ln(x^2 + y^2)$ has a particular solution that passes through $(e, 2)$. Use of the approximation formula gives $y_1 = 2.4$. Determine the value that was used for the step size. Use this value to find y_2 and y_3.
Solution:
We are given the starting point $(x_0, y_0) = (e, 2)$ and the first iteration value $y_1 = 2.4$. First, solve for the step size Δx:

$$\begin{aligned}
y_1 &= y_0 + \Delta x \cdot \ln(x_0^2 + y_0^2) \\
2.4 &= 2 + \Delta x \cdot \ln(e^2 + 2^2) \\
0.4 &= \Delta x \cdot \ln(e^2 + 4) \implies \Delta x = \frac{0.4}{\ln(e^2 + 4)} \approx 0.1644
\end{aligned}$$

Using $\Delta x \approx 0.1644$, we compute the next iterations (rounding to four decimal places):

$$x_1 = x_0 + \Delta x \approx 2.7183 + 0.1644 = 2.8827$$
$$y_2 = y_1 + \Delta x \cdot \ln(x_1^2 + y_1^2) \approx 2.4 + 0.1644 \cdot \ln((2.8827)^2 + (2.4)^2) \approx 2.4 + 0.1644(2.6441) \approx 2.8347$$

$$x_2 = x_1 + \Delta x \approx 2.8827 + 0.1644 = 3.0471$$
$$y_3 = y_2 + \Delta x \cdot \ln(x_2^2 + y_2^2) \approx 2.8347 + 0.1644 \cdot \ln((3.0471)^2 + (2.8347)^2) \approx 2.8347 + 0.1644(2.8519) \approx 3.3035$$

The step size used was $\Delta x \approx 0.1644$. The subsequent values are $y_2 \approx 2.835$ and $y_3 \approx 3.304$.